Calculus Facts

Limits
To evaluate substitute in the value or use graphical, symbolic methods to evaluate

Continuity
$f(x)$ is continuous at $x = a$ if

1. $\lim_{x\to a} f(x)$ exists (is finite)
2. $f(a)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

$f(x)$ is continuous on $[a,b]$ if it is continuous at every point in $[a,b]$.

Intermediate Value Theorem
If $y = f(x)$ is continuous on $[a,b]$ and $f(a) < w < f(b)$, then there exists c between a and b such that $f(c) = w$.

Extreme Value Theorem
If $y = f(x)$ is continuous on $[a,b]$ then f has both a maximum and a minimum on $[a,b]$.

Differentiability
may not be differentiable @ a if:

- discontinuous at a
- sharp turn at a
- cusp at a
- vertical tangent at a

Formal Def. Of Derivatives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{b\to a} \frac{f(b) - f(a)}{b-a}$$

Derivative Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d(cu)}{dx} = c\frac{du}{dx}$
3. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
4. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$
5. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
6. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
7. $\frac{d(f(g(x)))}{dx} = f'(g(x)) \cdot g'(x)$
8. $\frac{d(\log_a u)}{dx} = \frac{1}{u \ln a}\frac{du}{dx}$
9. $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$
10. $\frac{d(e^u)}{dx} = e^u\frac{du}{dx}$
11. $\frac{d(a^u)}{dx} = a^u \ln a\frac{du}{dx}$
12. $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$
13. $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$
14. $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$
15. $\frac{d(\cot u)}{dx} = -\csc^2 u\frac{du}{dx}$
16. $\frac{d(\sec u)}{dx} = \sec u \cdot \tan u\frac{du}{dx}$
17. $\frac{d(\csc u)}{dx} = -\csc u \cdot \cot u\frac{du}{dx}$
18. $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
19. $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
20. $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$
21. $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
22. $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$
23. $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

Derivatives of Inverse Functions
If f and g are inverses

$$f'(x) = \frac{1}{g'(f(x))}$$

Analysis of Curves
$f' > 0 \Rightarrow f$ increasing
$f' < 0 \Rightarrow f$ decreasing
$f' = 0$ *or und.* $\Rightarrow$ critical pts
possible f max, min,or flat
$\Rightarrow$ look for sign change
$f'' > 0 \Rightarrow f$ concave up
$f'' < 0 \Rightarrow f$ concave down
If f'' changes sign $\Rightarrow$ inflct. point

- max and min may occur at critical pts or endpts of closed intervals

Calculus Facts

Average Rate of Change/Slope

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

Instantaneous rate of change at "a" $= f'(a)$

Tangent Line,
Linear Approximation
$f(x) \approx f(a) + f'(a)(x - a)$
for x close to a.

Optimization
1. Draw a picture and label
2. Write eq. using 1 variable
3. Take Derivative
4. Find max or min

Velocity, Speed, Acceleration for motion along a line

position = $x(t)$ or $s(t)$
velocity = $v(t) = x'(t)$
acceleration =
$a(t) = v'(t) = x''(t)$
velocity = $\int a(t)\,dt$
position = $\int v(t)dt$
speed = $|v(t)|$
displacement = $\int_a^b v(t)\,dt$
total distance = $\int_a^b |v(t)|\,dt$
average velocity =
$$\frac{\text{final position} - \text{initial position}}{\text{total time}}$$

Mean Value Theorem
If $y = f(x)$ is **continuous** on $[a,b]$ and **differentiable** on (a,b) then there exists as secant line from a to b with the same slope as the tangent line at some c between a and b:
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Related Rates
1. Draw & label picture for things that change with time.
2. Write an equation that relates the variables.
3. Take the derivative implicitly with respect to time.
4. Substitute values for known quantities.
5. Attach units to answer.

Integration

The partition of f(x) on [a,b] given by $\lim_{x\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$

Left-hand Riemann Sums (LRAM)
$$\sum_{k=0}^{n-1} f(x_k)\Delta x,\ \ \Delta x = \frac{b-a}{n}$$

Right-hand Riemann Sums (RRAM)
$$\sum_{k=1}^{n} f(x_k)\Delta x,\ \ \Delta x = \frac{b-a}{n}$$

Midpoint Riemann Sums (MRAM)
$$\sum_{k=0}^{n-1} f(x_k + \frac{\Delta x}{2})\Delta x,\ \ \Delta x = \frac{b-a}{n}$$

Definite Integrals
$$\sum_{n=1}^{\infty} f(x_n)\Delta x = \int_a^b f(x)dx$$

Trapezoidal Rule
$$\int_a^b f(x)dx \approx \frac{1}{2}\left(\frac{b-a}{n}\right)\left(y_0 + 2y_1 + ... + 2y_{n-1} + y_n\right)$$

Interval Addition
$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Fundamental Theorem of Integral Calculus

$\int_a^b f(x)\,dx = F(b) - F(a)$,
where $F(x) = \int f(x)\,dx$

$$\frac{d}{dx}\int_a^u f(t)\,dt = f(u)\frac{du}{dx}$$

Integration Formulas/Rules

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. sum / difference
$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

4. Domination: if $f(x) > g(x)$ on [a,b], then
$\int_a^b f(x)dx > \int_a^b g(x)dx$

5. $\int du = u + C$

6. $\int a\,du = a\int du$

7. $\int u^n du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

8. $\int \frac{1}{u} du = \ln|u| + C$

9. $\int e^u du = e^u + C$

10. $\int a^u du = \frac{a^u}{\ln a} + C$

11. $\int \sin u\,du = -\cos u + C$

12. $\int \cos u\, du = \sin u + C$

13. $\int \tan u\, du = \ln|\sec u| + C$

14. $\int \cot u\, du = -\ln|\csc u| + C$

15. $\int \sec u\, du = \ln|\sec u + \tan u| + C$

16. $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

17. $\int \sec^2 u\, du = \tan u + C$

18. $\int \csc^2 u\, du = -\cot u + C$

19. $\int \sec u \cdot \tan u\, du = \sec u + C$

20. $\int \csc u \cdot \cot u\, du = -\csc u + C$

21. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

22. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

23. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

24. $\int \ln x dx = x \ln x - x + C$

Integration by parts

$\int u\, dv = uv - \int v\, du$

Tabular method works if u is successively differentiable and goes to zero, while v is easy to integrate repeatedly.

Integration by u-substitution
Usually let u be the more complex part of the integrand.

Trig substitutions

for $a^2 - u^2$ use:
$u = a\sin\theta, \quad du = a\cos\theta\, d\theta$

for $a^2 + u^2$ use:
$u = a\tan\theta, \quad du = a\sec^2\theta\, d\theta$

for $u^2 - a^2$ use:

$u = a\sec\theta, \quad du = a\sec\theta\tan\theta\, d\theta$

Average Value of a Function

If f is integrable on [a,b] then:

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

Mean Value for Definite Integrals
There is at least one x value (c) that assumes the average.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

Solving Differential Equations (Initial Value Problems
Separate variables, integrate, apply initial conditions

Slope Fields
Given a differential equation, calculate slopes by substituting coordinates in for x & y.

Ex: $\frac{dy}{dx} = y - x$, sub coordinates for x&y

Max/min rule
If M is the abs. Max on [a,b] and m is abs min on [a,b] then:

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Euler's Method

Given $\frac{dy}{dx} = f(x, y)$ & solution passes through (x_0, y_0),
$x_{new} = x_{old} + \Delta x$ then:
$y_{new} = y_{old} + f(x_0, y_0) \cdot dx$

Exponential Growth/Decay

if $\frac{dy}{dt} = ky$ then $y = y_0 e^{kt}$

- population is changing at a rate proportional to the amt. present
- continuous compounding growth

Newton's law of cooling

$T - T_s = (T_0 - T_s)e^{-kt}$
T – temp at time t
T_s - surrounding temp
T_0 - temp at t = 0

Logistic Growth

if $\frac{dy}{dt} = kP(M - P)$ then

$P = \frac{M}{1 + Ae^{-(Mk)t}}$, $A = \frac{M - P_0}{P_0}$

P = population
M = carrying capacity (max pop)
k = growth constant
A = constant found using the initial condition
Integrate with partial fractions.

Hooke's Law
The force it takes to stretch or compress a spring x units from its natural length is a constant times x. F = kx

Calculus Facts

Partial Fractions

$$\int \frac{dx}{(x-a)(x-b)} = \int \left(\frac{A}{(x-a)} + \frac{B}{(x-b)} \right) dx$$

Area between 2 curves

$\int_a^b \left(f(x) - g(x) \right) dx$ where $f(x) > g(x)$ meaning

$\int_a^b \left(top - bottom \right) dx$ or

$\int_c^d \left(right - left \right) dy$

L'Hopital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms

Volumes of Revolution

Vertical Cross Sections:

disks $V = \int_a^b \pi r^2 dx$

washers $V = \int_a^b \left(\pi R^2 - \pi r^2 \right) dx$

shells $V = \int_a^b \left(2\pi r h \right) dx$

cross sections

$= \int_a^b (area\ of\ \ face) dx$

Horizontal Cross Sections:
replace dx with dy

Arclength

x-y plane: $\int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2}\, dx$

or $\int_c^d \sqrt{1 + \left(\frac{dx}{dy} \right)^2}\, dy$

Parametric: $\int_a^b \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}\, dt$

Polar: $\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2}\, d\theta$

Polar functions

Area = $\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$

Area between 2 curves:

$\frac{1}{2} \int_{\theta_1}^{\theta_2} \left(r_1^2 - r_2^2 \right) d\theta$

Slope = $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$; $x = r\cos\theta$, $y = r\sin\theta$

Or $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

Parametric and Vector Functions

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \left(\frac{dy}{dt} \right) \bullet \left(\frac{dt}{dx} \right)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{dy'}{dx}}{\frac{dx}{dt}}$$

Vectors

Speed = |velocity| = $\sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}$

Velocity vector = $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

Acceleration vector = $\left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$

position:
$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

velocity:
$\mathbf{v} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

acceleration:
$\mathbf{a} = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$

speed =
$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

total distance =
$\int_{t_0}^{t_f} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$

Infinite Series

Taylor Series

If a function is "smooth" at x = a, then f can be approximated by the nth degree polynomial:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ..$$

$$\approx \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Maclaurin series if $a = 0$.

Common Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} +$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(all real x)}$$

Calculus Facts

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + $$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{(all real x)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + $$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{(all real x)}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + $$
$$= \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + $$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad \left(-1 < x \le 1\right)$$

Convergence Tests

nth term test:

if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges

Geometric series:

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$ and a is the first term

converges only for $-1 < r < 1$

***p*-series**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if p>1, diverges if $p \le 1$

Integral test: $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a_n \, dn$

a_n is continuous, positive, and decreasing then either both converge or both diverge

Ratio Test, nth Root test

$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, $L = \lim_{n\to\infty} \sqrt[n]{a_n}$

$L < 1$, series converges
$L > 1$, series diverges
$L = 1$, test fails – try another test

Comparison test:

If $a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $a_n > b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Alternating Series Test:

If 1. signs strictly alternate
2. $|a_{n+1}| < |a_n|$
3. $a_n \to 0$ as $n \to \infty$

Series converges and remainder of $\sum_{k=1}^{n} a_k < |a_{n+1}|$

Absolute Convergence:

if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges absolutely

Conditional Convergence:

$\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges.

Lagrange error bounds (Taylor's Theorem with Remainder)

Gives a formula for the error involved in approximating the polynomial over the given interval.

$$\left| f(c) - P_n(f,a)(c) \right| < R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \left(x-a\right)^{n+1}$$

Calculus Facts

Trigonometry

θ	sin θ	cos θ	tan θ
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undef

Pythagorean Identities

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$\sec^2\theta-1=\tan^2\theta$

$1+\cot^2\theta=\csc^2\theta$

Double Angle Identities

$\sin 2\theta=2\sin\theta\cos\theta$

$\cos 2\theta=\cos^2\theta-\sin^2\theta$

$=2\cos^2\theta-1$

$=1-2\sin^2\theta$

Power Reducing Identities

$\sin^2\theta=\dfrac{1-\cos 2\theta}{2}$

$\cos^2\theta=\dfrac{1+\cos 2\theta}{2}$

Odd/Even

$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=\cos\theta$

$\tan(-\theta)=-\tan\theta$

Reciprocal Identities

$\sec x=\dfrac{1}{\cos x}$

$\csc x=\dfrac{1}{\sin x}$

Geometry

Circle

$A=\pi r^2,\ C=2\pi r$

Sphere

$SA=4\pi r^2,\ V=\dfrac{4}{3}\pi r^3$

Right Circular Cylinder

$SA=2\pi r^2+2\pi rh,\ V=\pi r^2 h$

Right Circular Cone

$V=\dfrac{\pi}{3}r^2 h$

Algebra

point-slope equation of a line

$y=m(x-x_1)+y_1$

Properties of exponents

$a^m\cdot a^n=a^{m+n}$

$\dfrac{a^m}{a^n}=a^{m-n}$

$a^{-m}=\dfrac{1}{a^m}$

$a^{\frac{m}{n}}=\sqrt[n]{a^m}$

Properties of logarithms

$\log_b(mn)=\log_b m+\log_b n$

$\log_b\dfrac{m}{n}=\log_b m-\log_b n$

$\log_b m^n=n\log_b m$

$b^{\log_b m}=m$

$\log_b b=1$

$\log_b 1=0$

$\log_b x=\dfrac{\ln x}{\ln b}$

$y=\log_b x$ is the inverse of $y=b^x$

Domain Restrictions

- even index radical ≥ 0
- $\log_b x,\ x>0$
- denominator $\neq 0$ causes Vertical Asymptotes (VA) or holes (factors that divide out)

Horizontal Asymptotes: HA

$\lim\limits_{x\to\pm\infty} f(x)=b$ then y = b is HA

(top heavy, bottom heavy, equal)

Intercepts

x-intercepts y = 0, solve for x

y-intercepts x = 0, solve for y

Analytic Geometry

Distance between 2 points

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Conic Sections

Parabola with vertex (h,k) and focal length $|a|$

$(x-h)^2=4a(y-k)$

opens up if $a>0$, down if $a<0$

$(y-k)^2=4a(x-h)$

opens right if $a>0$, left if $a<0$

graphing form:

$y=a(x-h)^2+k$

Circle with center (h,k) and radius r

$(x-h)^2+(y-k)^2=r^2$

Ellipse with center (h,k)

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

semimajor axis = larger of a and b semiminor axis = smaller of a and b

Hyperbola centered on (*h*,*k*)

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

opens horizontally

$$\frac{(y-h)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

opens vertically

Order of Magnitude

1. factorial
2. exponential
3. power
4. logarithmic

Polar Coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}\frac{y}{x}$

add π to θ if in Quad II or III

Circles

$r = k \quad r = a\cos\theta \quad r = a\sin\theta$

Leafed Roses

$r = a\sin(n\theta) \quad r = a\cos(n\theta)$

If *n* is even, period = 2π, number of leaves = 2*n*. If *n* is odd, period = π, number of leaves = *n*.

Cardioids

$r = a(1\pm\cos\theta) \quad r = a(1\pm\sin\theta)$

period = 2π

Limacons

$r = a\pm b\sin\theta \quad r = a\pm b\cos\theta$

period = 2π

Lemniscates

$r^2 = a\cos 2\theta \quad r^2 = a\sin 2\theta$

period = $\frac{\pi}{2}$