9.5b Alternating Series, Checking Endpoints

Alternating Series

An alternating series is a series whose terms are alternately positive and negative

Example: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

Alternating Series Test

If $a_n>0$, then an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$

converges if both of the following conditions are met:

a). series must be alternating.

1) $\lim_{n\to\infty} a_n=0$

2) $\{a_n\}$ is a decreasing sequence; that is, $a_{n+1}<a_n$ for all n.

Read Note

Ex. Determine whether the following series converges or diverges.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2n-1}$ alt

$1, \frac{2}{3}, \frac{3}{5}$

✓ 1. alt

2. $a_n \to 0$ Find a new test

✓ 3. dec. in mag.

$\lim_{n\to\infty} a_n \neq 0$ Diverging

n^{th} term test

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

✓ 1. alt.

✓ 2. $\lim_{n\to\infty}\frac{1}{n}=0$

✓ 3. dec. in mag.

Converging by A.S.T.

Alternating Series Remainder (truncation error)

Suppose a series has terms that are alternating, decreasing in magnitude, and having a limit of 0. If the series has a sum S, then

$|R_n| = |S - S_n| < a_{n+1}$, where S_n is the nth partial sum of the series.

$|E|$

In other words, if the three conditions are met, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will be bounded by the first truncated term, a_{n+1}.

Ex.
Find a bound for the truncation error after 99 terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1} (.1)^n$$

$|E| < .1^{100}$

✓ alt

✓ dec. in mag

✓ $\lim_{n\to\infty} (.1)^n = 0$

Ex. Approximate the sum, S, of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using its first four terms, and explain why your estimate differs from the actual value by less than $\frac{1}{100}$. Then use your results to find an interval in which S must lie.

$$S_4 = 1 + \frac{(-1)}{2!} + \frac{1}{3!} - \frac{1}{4!} \quad \frac{1}{5!}$$

$\cancel{1} - \frac{1}{2} + \frac{1}{6} - \frac{1}{24}$

$\frac{1}{2} + \frac{1}{6} - \frac{1}{24}$

$\frac{15}{24}$

$$|E| < \frac{1}{5!}$$

$$|E| < \frac{1}{120} < \frac{1}{100}$$

$$\frac{15}{24} - \frac{1}{120} < S < \frac{15}{24} + \frac{1}{120}$$

$$\frac{74}{120} < S < \frac{76}{120}$$

Definitions:

$\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges

Determine whether the given alternating series converges or diverges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST

✓ alt
✓ dec in mag.
✓ $a_n \to 0$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right|$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$p = \frac{1}{2} < 1$

Diverges

Conditionally Convergent

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$ converges by AST

✓ alt.
✓ dec. in mag.
✓ $a_n \to 0$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{3^n}\right|$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$r = \frac{1}{3} < 1$

Converges by Geom. Series Test

Absolutely Convergent

Find the interval of convergence for the following series.
Be sure to check the endpoints.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{2n}$$

$$\frac{x}{2(n+1)} \cdot \frac{2n}{x}$$

$$\lim_{n\to\infty} \frac{x^2\, 2n}{2n+2} = x^2$$

$x^2 = 1 \quad x = \pm 1$

$$|x^2| < 1$$

$$-1 < x^2 < 1$$

$$-1 < x < 1$$

Check Endpts. plug it into the Σ

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^{2n}}{2n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}$$

Converges

✓ alt
✓ dec. in mag.
✓ $a_n \to 0$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^{2n}}{2n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{3n+1}}{2n}$$

Converges

✓ alt
✓ dec in mag.
✓ $a_n \to 0$

$$-1 \le x \le 1$$

$$\sum_{n=0}^{\infty}\frac{(10x)^n}{n!}$$

$$\frac{(10x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(10x)^n}$$

$$(n+1)\,n!$$

$$\lim_{n\to\infty}\frac{10x}{n+1} = 0$$

$\mathbb{R} \qquad (-\infty, \infty)$

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2n}$$

$$\lim_{n\to\infty} \frac{(x-3)^{\cancel{n}+1}}{2(n+1)} \cdot \frac{2n}{\cancel{(x-3)^n}}$$

$= x-3$

$|x-3| < 1$

$-1 < x-3 < 1$

$2 \leq x < 4$

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{2n}$$

$$\frac{(-1)^n}{2n}$$

√ alt
√ dec in mag.
√ $a_n \to 0$

converge

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{2n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

$$\int_1^{\infty} \frac{1}{2n}\,dn$$

$$\frac{1}{2}\int_1^{\infty} \frac{1}{n}\,dn$$

$$\lim_{b\to\infty} \frac{1}{2} \ln n \Big|_1^b$$

$$\lim_{b\to\infty} \frac{1}{2}(\ln b - \ln 1)$$

diverges