Warm-Up

The function f has derivatives of all orders for all real numbers x. Assume that $f(2)=6$, $f'(2)=4$, $f''(2)=-7$, $f'''(2)=8$.

(a) Write the third-degree Taylor polynomial for f about $x=2$, and use it to approximate $f(2.3)$. Give three decimal places. 6.921

$$P_3(x) = 6 + 4(x-2) - \frac{7(x-2)^2}{2!} + \frac{8(x-2)^3}{3!}$$

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 9$ for all x in the closed interval $[2, 2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a, b]$ such that $a \le f(2.3) \le b$. Give three decimal places.

(c) Could $f(2.3)$ equal 6.922? Explain why or why not.

(d) Could $f(2.3)$ equal 6.927? Explain why or why not.

$$|E| \le \frac{M(\overset{2.3}{x}-2)^4}{4!}$$

$$\boxed{|E| \le .0030375}$$

$$6.918 \le 6.921 \le 6.9230375$$

9.4

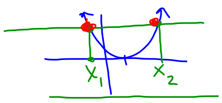
23. $$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n} = \frac{((x-1)^2)^n}{4^n} = \left(\frac{(x-1)^2}{4}\right)^n$$

$$r = \frac{(x-1)^2}{4}$$

$$-1 < \frac{(x-1)^2}{4} < 1 \qquad |(x-1)^2| < 4$$

$$-4 < (x-1)^2 < 4$$

$$-1 < x < 3$$



$$(x-1)^2 = 4$$

$$x-1 = \pm 2$$

$$x = -1, 3$$

$$S = \frac{1}{\frac{4}{4} - \frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2}$$

31. $\sum_{n=1}^{\infty} \frac{n^2-1}{2^n}$

$$\frac{(n+1)^2-1}{2^{\cancel{n}+1}} \cdot \frac{\cancel{2^n}}{n^2-1}$$

$$\lim_{n\to\infty} \frac{\cancel{(n+1)^2-1}}{2\cancel{(n^2-1)}} = \frac{1}{2} < 1$$

$\frac{1}{2}$

Converges by Ratio Test

35. $\sum_{n=0}^{\infty} n^2 e^{-n}$

$$\frac{n^2}{e^n}$$

$$\frac{(n+1)^2}{e^{\cancel{n}+1}} \cdot \frac{\cancel{e^n}}{n^2}$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{n^2 \cdot e} = \frac{1}{e} < 1$$

Converges by Ratio Test

39.

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^n}$$

$$\left(\frac{-2}{3}\right)^n \qquad r = \left|\frac{-2}{3}\right| < 1$$

converges by Geom. Test

9.5a More Tests for Convergence

Integral Test

If f is positive, continuous, and decreasing for $x \ge 1$ and $a_n = f(x)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or diverge.

Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\int_1^{\infty} \frac{n}{n^2+1}\,dn \qquad u = n^2+1 \qquad \frac{du}{dn} = \cancel{2}n\,dn$$

$$\frac{1}{2}\int \frac{\cancel{n}}{u}\frac{du}{\cancel{2n}} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}\lim_{b\to\infty} \ln(n^2+1)\Big|_1^b$$

$$\frac{1}{2}\left(\lim_{b\to\infty} \ln(b^2+1) - \ln(2)\right)$$

$$= \infty$$

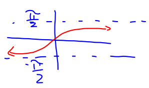
Diverges by Int. Test

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\int_1^{\infty} \frac{1}{n^2+1}\,dn$$

$$\lim_{b\to\infty}\left(\tan^{-1} n\Big|_1^b\right)$$

$$\lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} 1\right)$$



$$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Converges by Int. Test

p-Series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$ is called a p-series, where p is a positive constant. For p = 1, the series

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$ is called the harmonic series.

Use the Integral test to try to figure out what values of p would cause $\sum_{n=1}^{\infty}\frac{1}{n^p}$ to converge.

p = 1 $\sum_{n=1}^{\infty}\frac{1}{n^1}$ $\int_1^{\infty}\frac{1}{n}\,dn = \lim_{b\to\infty}(\ln b - \ln 1)$

$= \infty$ diverges

p = 2 $\sum \frac{1}{n^2}$ $\int_1^{\infty} n^{-2}\,dn = \lim_{b\to\infty} \left.\frac{-1}{n^1}\right|_1^b$

$\lim_{b\to\infty} \cancel{\frac{-1}{b}}^{\,0} - \left(\frac{-1}{1}\right)$

$0+1=1$

Converges

$p=\frac{1}{2}$ $\sum \frac{1}{n^{\frac{1}{2}}}$

$\int_1^{\infty} n^{-\frac{1}{2}}\,dn$

$\lim_{b\to\infty} \left. 2n^{\frac{1}{2}}\right|_1^b$

$2b^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}$

$= \infty$ diverges

p-Series Test

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

a) converges if p > 1

b) diverges if p < 1

c) diverges if p = 1

Determine whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1}\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$$

$p > 1$ converges

Limit Comparison Test

Suppose $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is both finite and positive. (This means a_n and b_n grow at the same rate!)

Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+10}{4n^5-n^3+7}$

$$\lim_{n\to\infty} \frac{\frac{n^4+10}{4n^5-n^3+7}}{\frac{1}{n}} \cdot \frac{n}{1} = \frac{1}{4}$$

a_n & b_n grow @ same rate

& $\frac{1}{n}$ diverges by p-series

$\therefore \frac{n^4+10}{4n^5-n^3+7}$ also div

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^3-2}}{\cancel{\frac{1}{n^3}}} \cdot \frac{n^3}{1} = 1$$

$\frac{1}{n^3-2}$ & $\frac{1}{n^3}$ grow @ same rate

$\frac{1}{n^3}$ converges by p-series

$\therefore \frac{1}{n^3-2}$ also converges by limit comp. test

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\cancel{\frac{1}{2^n}}} \cdot \frac{2^n}{1} = 1$$

$\left(\frac{1}{2}\right)^n$ $r = \frac{1}{2} < 1$ converges by geom test

$$-6(y-2)^{-1}$$

$$10x-6\left(-1(y-2)^{-2}\frac{dy}{dx}\right)$$

$$10x+\frac{6}{(y-2)^2}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=10x+\frac{6}{(y-2)^2}\left(5x^2-\frac{6}{y-2}\right)$$

b. $-4+\frac{6(x+1)^1}{1!}-\frac{9(x+1)^2}{2!}$