8.4b Improper Integrals

Comparison Test

if $0 \le f(x) \le g(x)$

and $\int_a^\infty f(x)dx \le \int_a^\infty g(x)dx$

and $g(x)$ converges then $f(x)$ also converges.

and $\int_a^\infty f(x)dx \le \int_a^\infty g(x)dx$

and $f(x)$ diverges

then $g(x)$ also diverges

$\int_1^\infty e^{-x^2}dx$

$\frac{1}{e^{x^2}} \le \frac{1}{e^x}$

$\int_1^\infty e^{-x}dx$

$\lim_{b\to\infty} \int_1^b e^{-x}dx$

$\lim_{b\to\infty} \left(-e^{-x} \Big|_1^b \right)$

$\lim_{b\to\infty} \left(\cancel{-e^{-b}}^{\,0} - (-e^{-1}) \right)$

$\frac{1}{e}$

because $\int_1^\infty e^{-x}dx$ converges to $\frac{1}{e}$

and $e^{-x^2} \le e^{-x}$ then $\int_1^\infty e^{-x^2}dx$

also converges to value $\le \frac{1}{e}$

Does the integral converge or diverge?

$$\int_1^\infty \frac{dx}{x^5+1}$$

$$\frac{1}{x^5+1} \le \frac{1}{x^5}$$

$$\int_1^\infty x^{-5}\,dx$$

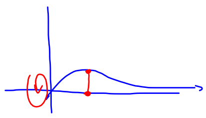
$$\lim_{b\to\infty}\left(\frac{x^{-4}}{-4}\Big|_1^b\right)$$

$$\lim_{b\to\infty}\left(\cancel{\frac{1}{-4b^4}}^{0} - \left(\frac{1}{-4(1)^4}\right)\right)$$

$\int_1^\infty \frac{1}{x^5+1}\,dx$ converges to a value $< \frac{1}{4}$

$\frac{1}{4}$

Find the volume of the solid obtained by revolving the curve about the x-axis: $y = xe^{-x}$ $0 \le x < \infty$



$\int \pi r^2 \, dx$

$$\pi \int_0^\infty \left(xe^{-x}\right)^2 dx$$

$$\pi \lim_{b\to\infty} \int_0^b x^2 e^{-2x}\, dx$$

x^2	e^{-2x}
$2x$	$+\ \frac{-e^{-2x}}{2}$
2	$-\ \frac{e^{-2x}}{4}$
0	$+\ \frac{-e^{-2x}}{8}$

$$\pi \lim_{b\to\infty}\left(-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\Big|_0^b\right)$$

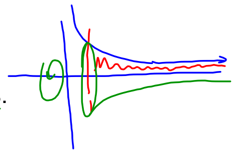
$$\pi \lim_{b\to\infty}\left(\cancel{-\frac{b^2e^{-2b}}{2}}^{0} - \cancel{\frac{be^{-2b}}{2}}^{0} - \cancel{\frac{e^{-2b}}{4}}^{0}\right) - \left(0 - 0 - \frac{e^0}{4}\right)$$

$\frac{1}{4}\pi$

Gabriel's Horn

Consider the region R in the fist quadrant bounded above by:

$y = \frac{1}{x}$ and on the left by $x = 1$



The region is revolved around the x-axis.

a. show the R has infinite area.

b. Find the volume of the solid.

a. $\int_1^{\infty} \frac{1}{x}\,dx$

$\lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$

$\lim_{b\to\infty} \left(\ln x \Big|_1^b \right)$

$\lim_{b\to\infty} \left(\ln b - \cancel{\ln 1} \right)$

$= \infty$

diverge

b. $\pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx$

$\pi \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx$

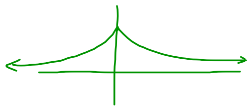
$\pi \lim_{b\to\infty} \left(-x^{-1} \Big|_1^b \right)$

$\pi \lim_{b\to\infty} \left(\cancel{-b^{-1}}^{\,0} - \left(-1^{-1}\right) \right)$

$= \pi$

8.4 a

21. $\int_{-\infty}^{\infty} e^{-|x|}\,dx$



$2 \left(\int_0^{\infty} e^{-x}\,dx \right)$

$2 \lim_{b\to\infty} \int_0^{b} e^{-x}\,dx$

$2 \lim_{b\to\infty} \left(-e^{-x} \Big|_0^b \right)$

$2 \lim_{b\to\infty} \left(\cancel{-e^{-b}}^{\,0} - \left(-e^{-0}\right) \right)$

$2(1) = 2$

27.

$$\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$$

$x(x+2)$

$$\lim_{b\to 0} \int_b^1 \frac{x+1}{(x^2+2x)^{\frac{1}{2}}}\,dx$$

$u = x^2 + 2x$

$du = 2x + 2\ dx$

$du = 2(x+1)\,dx$

$$\int \frac{\cancel{x+1}}{u^{\frac{1}{2}}} \cdot \frac{du}{2(\cancel{x+1})}$$

$$\frac{1}{2} \int u^{-\frac{1}{2}}\,du$$

$$\frac{1}{\cancel{2}} \cdot \cancel{2} u^{\frac{1}{2}}$$

$$\lim_{b\to 0} \left((x^2+2x)^{\frac{1}{2}} \Big|_b^1 \right)$$