10.1 Parametric Equations

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equations that describe motion in terms of time because they have an additional variable (the parameter) -where an item is at a particular moment in time

there are 2 parts to a parametric equation - in terms of time

$$x = f(t)$$
$$y = g(t)$$

this is one set of equations

t is the parameter

plot points as (x, y)

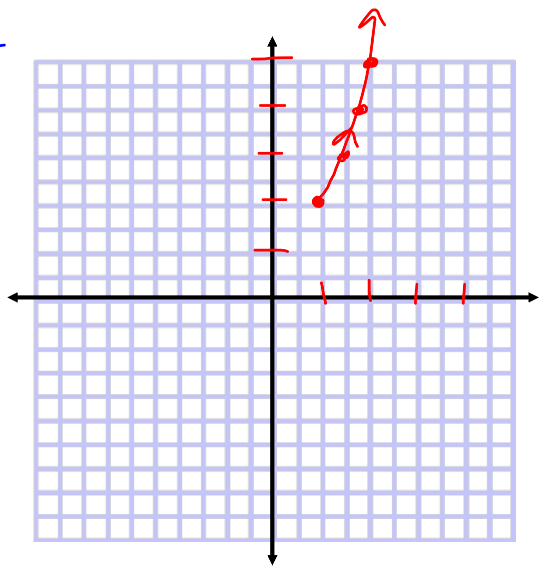
Make a table of values, sketch the curve, indicate direction. Then eliminate the parameter.

$$x = \sqrt{t+1}$$
$$y = t \cancel{+2}$$
-2

t	0	1	2	3
x	1	$\sqrt{2}$	$\sqrt{3}$	2
y	2	3	4	5

$x = \sqrt{(y-2)+1}$

$x = \sqrt{y-1}$



Make a table of values, sketch the curve, indicate direction. Then eliminate the parameter.

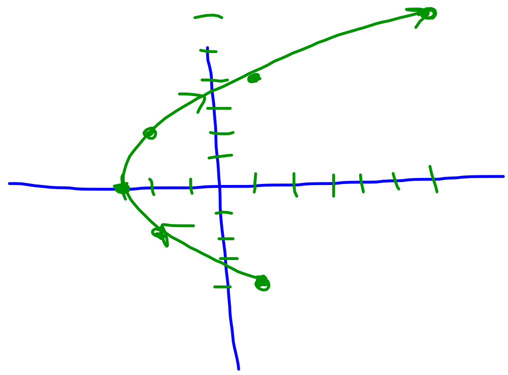
$x = t^2 - 3$

$\frac{y}{2} = \cancel{2}t$

$-2 \le t \le 3$

t	-2	-1	0	1	2	3
x	1	-2	-3	-2	1	6
y	-4	-2	0	2	4	6

$x = \left(\frac{y}{2}\right)^2 - 3$



Eliminate the parameter in the following equation:

$x = 3\cos t$

$\left(\frac{y}{4}\right)^2 = \cancel{4}(\sin t)^2$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{y^2}{16} + \frac{x^2}{9} = 1$

$y = 2\sec t$ $\tan^2 t + 1 = \sec^2 t$

$x = 5\tan t$

$$\left(\frac{x}{5}\right)^2 + 1 = \left(\frac{y}{2}\right)^2$$

$$\frac{y^2}{4} - \frac{x^2}{25} = 1$$

Parametric Equations and Calculus

If a smooth curve is represented by parametric equations, then the slope of the curve at a point (x, y) is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{where} \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Ex. 1 (Noncalculator)

Given the parametric equations $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$2t^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t-2}{t^{-\frac{1}{2}}} = t^{\frac{1}{2}}(6t-2)$$

$$= 6t^{\frac{3}{2}} - 2t^{\frac{1}{2}}$$

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{9t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{-\frac{1}{2}}} = t^{\frac{1}{2}}\left(9t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)$$

$$= 9t - 1$$

$$\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}} = \frac{9}{t^{-\frac{1}{2}}} = 9t^{\frac{1}{2}}$$

Ex. 2 (Noncalculator)

Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$ write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

pt. $\left(4\left(-\frac{\sqrt{2}}{2}\right), 3\frac{\sqrt{2}}{2}\right)$

$\left(-2\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$

$$\frac{dy}{dx} = \frac{3\cos t}{-4\sin t}\Bigg|_{t=\frac{3\pi}{4}}$$

slope:

$$\frac{3\left(-\frac{\sqrt{2}}{2}\right)}{-4\left(\frac{\sqrt{2}}{2}\right)} = \frac{3}{4}$$

$$y = \frac{3}{4}\left(x + 2\sqrt{2}\right) + \frac{3\sqrt{2}}{2}$$

Ex 3 (Noncalculator)

Find all points of horizontal and vertical tangency given the parametric equations $x = t^2 + t,\ \ y = t^2 - 3t + 5$.

Slope: $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 0$ horizontal $\quad \frac{dy}{dt} = 0 \quad \frac{dx}{dt} \neq 0$

vertical und. =

$\frac{dx}{dt} = 0$

$\frac{dy}{dt} = 2t - 3 = 0$

$\frac{dx}{dt} = 2t + 1 = 0$

$t = \frac{3}{2}$

$\left(\frac{3}{2}\right)^2 + \frac{3}{2} \quad \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5$

$\left(\frac{15}{4}, \frac{11}{4}\right)$

$t = -\frac{1}{2}$

$\left(-\frac{1}{2}\right)^2 + -\frac{1}{2},\ \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 5$

$\left(\frac{-1}{4}, \frac{27}{4}\right)$

Ex 4

Sketch the graph, indicate the direction, find the lowest point, and find any points of inflection.

$x = t^2 - 3 \qquad 0 \le t \le \pi$

$y = -4\sin t$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	-3	$\frac{\pi^2}{16} - 3$	$\frac{\pi^2}{4} - 3$	$\frac{9}{16}\pi^2 - 3$	$\pi^2 - 3$
y	0	$-2\sqrt{2}$	-4	$-2\sqrt{2}$	0

$(-3, 0)\ (-2.383, -2.828)\ (-.533, -4)\ (2.551, -2.88)$

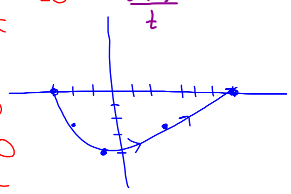
min.

$\frac{dy}{dx} = \frac{-\cancel{4}^{\,2}\cos t}{\cancel{2}t} = 0 \quad = -\frac{2\cos t}{t}$

$-4\cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}$



$\frac{d^2y}{dx^2} = \dfrac{\frac{t(2\sin t) - (-2\cos t)}{t^2}}{\cancel{2t}} \cdot \frac{1}{2t}$

$\frac{2t\sin t + 2\cos t}{2t^3} = 0$

$2t\sin t + 2\cos t = 0$

$t\sin t + \cos t = 0$

Finish w/ calc.

Arc Length

from t = a to t = b, or distance traveled by the particle from a to b

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Ex. 5 (Noncalculator)

Set up an integral expression for the arc length of the curve given by the parametric equations $x = t^2 + 1,\ \ y = 4t^3 - 1,\ \ 0 \le t \le 1.$

Do not evaluate.

$$\int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\ dt$$

Ex 6: Find the arc length:

$x = 2\cos^3 t$ $\quad 0 \le t \le \pi$

$y = 2\sin^3 t$

$$\int_0^\pi \sqrt{(6\cos^2 t \cdot -\sin t)^2 + (6\sin^2 t \cdot \cos t)^2}\, dt$$

$$\int_0^\pi \sqrt{36\cos^4 t \cdot \sin^2 t + 36\sin^4 t \cdot \cos^2 t}\, dt$$

$$\int_0^\pi \sqrt{36\cos^2 t \sin^2 t \overset{1}{\cancel{(\cos^2 t + \sin^2 t)}}}\, dt$$

$$\int_0^\pi 6\cos t \sin t \, dt \qquad \begin{array}{l} u = \sin t \\ du = \cos t\, dt \end{array}$$

$$6\int u\, du$$

$$6\left(\frac{u^2}{2}\right)$$

$$6\,\frac{\sin^2 t}{2}\Big|_0^\pi$$

$$0 - 0 = 0$$